Introduction

A noticeable and practicable scheme of channel coordination is the quantity flexibility contracts. It can be emphasized that quantity flexibility contracts can share the risk and cost of market demand uncertainty and yield more potential profitability for members within a supply chain. In view of its increasing importance, quantity flexibility contracts have been studied extensively and various flexibility provisions have been developed for particular application circumstances in recent years. Fisher and Raman (1996) analyzed how quantity flexibility promises made by retailers during two periods influence supplier production operations. Tsay (1999) argued that the quantity flexibility contract enables manufacturer and retailer to share the cost of market demand uncertainty. Tsay and Lovejoy (1999) then extended the model to more complex settings with multiple echelons, multiple periods and demand forecast updates and applied rolling-horizon planning.

Milner and Rosenblatt (2002) examined a two-period supply chain contract scenario and presented an approach for determining initial order quantity and optimal adjustment policy for second period. Lian and Deshmukh (2009) also explored another type of rolling-horizon planning contracts. Chan and Chan (2006) transformed the supply chain problem into a distributed constraint satisfaction problem and the coordinating mechanism used in inventory management is developed for the distributed supply chain with quantity flexibility. Yazlali and Erhun (2007) simultaneously explored multiple supply and quantity flexibility contract issues and identified their similarities. Sethi, Yan, and Zhang (2004) presented another quantity flexibility contract that considered the existence of the spot market. Wu (2005) studied a quantity flexibility contract in which the retailer initially provides the manufacturer with market information and later makes purchasing quantity decisions. Barnes-Schuster, Bassok, and Anupindi (2002) proposed taking advantage of options on capacity offered by the supplier as the instrument of quantity flexibility.

In comparison with existing literature involving quantity flexibility contracts, there are three remarkable distinctions existing between this study and previous researches. Firstly, in the vast majority of designs of quantity flexibility contract, order quantities only allow for replenishing but do not allow for lessening in the later period. The quantity flexibility contracts proposed in this study can suitably tackle this problem by means of flexible arrangement of immediate supplements and permissible returns. Secondly, in this study buyer and seller can arbitrarily negotiate and choose the desired range of quantity flexibility while two parities accept each other. Lastly and most importantly, this paper would like to emphasize that this study introduces originally the real option-based conception and approach to develop the more precise and reasonable assessing method involving quantity flexibility contracts.

Modeling and Assessing Quantity Flexibility Contracts

This study introduces the Ito process, which is widely applied to model the price shift behavior of highly volatile financial assets, to materialize demand shift by stochastic differential equation. The formal Ito process for random variable $X_t$, signifying market demand during period $t$, can be formulated as follows:

$$dX_t = \mu(X_t, t) dt + \sigma(X_t, t) dz_t .$$

(1)

The variable $X_t$ is assumed here to exhibit log-normal distribution and thus variable $z_t$ satisfies the standard Wiener process, and its volatility can be expressed as $dz_t = \sqrt{dt}$. If $X$ represents the demand variable of a given product and the diffusion component is approximately stationary (independent of time) which is expectedly held for the demand process over a relatively finite horizon, a specific form $\mu(X, t) = \mu X_t$ and $\sigma(X, t) = \sigma X_t$ can be obtained. As a result of specific form, Eq. (1) can then be reformulated as follows:

$$\frac{dX_t}{X_t} = \mu dt + \sigma dz_t .$$

(2)

Eq. (2) can then be transformed into the following well-known expression through the Taylor expansion.

$$d \ln X_t = \left( \mu - \frac{1}{2} \sigma^2 \right) dt + \sigma dz_t .$$

(3)
Integrating both sides of Eq. (3) obtains the following discrete-time model:

\[
\ln X_{t+\Delta t} - \ln X_t = \left( \mu - \frac{1}{2} \sigma^2 \right) \Delta t + \sigma \varepsilon \sqrt{\Delta t}, \text{ or } (4)
\]

where

\[
\Delta t = \text{length of period},
\mu = \text{expected growth rate of demand variable } X,
\sigma = \text{standard deviation of growth rate of demand variable } X,
\varepsilon = \text{a standard normal random variable; i.e., } \varepsilon \sim N(0,1).
\]

Eq. (5) shows that the expected value and variance of \(\ln X_{t+\Delta t}\) are

\[
E[\ln X_{t+\Delta t}] = \ln X_t + \left( \mu - \sigma^2 / 2 \right) \Delta t \quad \text{and} \quad \text{Var}[\ln X_{t+\Delta t}] = \sigma^2 \Delta t,
\]

respectively.

A quantity flexibility contract can be regarded as the composite of an ordinary order contract and an additional option on order quantity that endows buyer with the right to enlarge or lessen the original order quantity through replenishing or returning portion of units within the appointed quantity flexibility interval prior to the end of upcoming selling season. The following symbols are defined.

\[
p = \text{sale net income per unit for the given product},
\]
\[
c = \text{purchase cost per unit for the given product},
\]
\[
X_0 = \text{actual market demand for the given product during previous selling season},
\]
\[
Q_0 = \text{original order quantity within the given quantity flexibility contract},
\]
\[
Q_1 = \text{appointed lower-bound order quantity within the given quantity flexibility contract},
\]
\[
Q_2 = \text{appointed upper-bound order quantity within the given quantity flexibility contract},
\]
\[
X_f = \text{market demand for the given product during the upcoming selling season},
\]
\[
r = \text{risk-free annual rate},
\]
\[
v = \text{expected salvage value or liquidating price per unit for remaining unsold units},
\]
\[
T = \text{duration of the upcoming selling season}.
\]

By using the foregoing symbols, the end-of-period value (exercise value) of an option on order quantity can be formulated as follows.

\[
B_T = \begin{cases} 
(p-c)(Q_2 - Q_1), & X_f \geq Q_2 \\
(p-c)(X_f - Q_1), & Q_0 < X_f < Q_2 \\
0, & X_f = Q_0 \\
(c-v)(Q_2 - X_f), & Q_1 < X_f < Q_0 \\
(c-v)(Q_2 - Q_0), & X_f < Q_1 \end{cases}
\]

Based on Eq. (6), the expected end-of-period value of an option on order quantity can be expressed as follows.

\[
E[B_T] = (p-c)(Q_2 - Q_0) \times P(X_f \geq Q_2) + (p-c)(X_f - Q_0) \times P(Q_0 \leq X_f < Q_2) + (c-v)(Q_0 - X_f) \times P(Q_0 \leq X_f < Q_0) + (c-v)(Q_2 - Q_0) \times P(X_f \leq Q_1). \quad (7)
\]

The probability function density of the demand variable \(Q_T\) can be expressed as follows:

\[
f(X_f) = \frac{1}{X_f \sigma \sqrt{T/2\pi}} \exp \left[ \frac{-(\ln X_f - E[\ln X_f])^2}{2\sigma^2 T} \right]. \quad (8)
\]

Right-hand first and second terms in Eq. (7) are jointly solved, and so do for right-hand third and fourth terms.

**Right-hand First and Second Terms**

1. **First Term**

\[
(p-c)(Q_2 - Q_0) \times P(X_f \geq Q_2) = (p-c)Q_2 \times P(X_f \geq Q_2) - (p-c)Q_0 \times P(X_f \geq Q_2) \quad (9)
\]

2. **Second Term**

\[
(p-c)(X_f - Q_0) \times P(Q_0 \leq X_f < Q_2) = (p-c)(X_f - Q_0) \times P(X_f \geq Q_0) - (p-c)X_f \times P(X_f \geq Q_2) + (p-c)Q_0 \times P(X_f \geq Q_2) \quad (10)
\]

By combining Eqs. (9) and (10) following expression can be generated:

\[
(p-c)(X_f - Q_0) \times P(X_f \geq Q_0) - (p-c)(X_f - Q_1) \times P(X_f \geq Q_1) = (p-c)(C_1 - C_2). \quad (11)
\]

Eq. (11) implicatively consists of two call options in which the exercise prices are \(Q_0\) and \(Q_2\) respectively and can be separately solved.
(1) \((X_t - Q_0) \times P(X_t \geq Q_0)\)

\[(X_t - Q_0) \times P(X_t \geq Q_0) = \int_{Q_0}^{\infty} (X_t - Q_0) \times f(X_t) \, dx \, e^{-\theta}\]  

(12)

This component can be algebraically solved and the procedure is shown in Appendix. After merging Eqs. (A.4) and (A.7), a close-form solution for first component of Eq. (11) can be worked out as follows:

\[C_1 = X_0 \exp(\mu T) \times N(d_{11}) - Q_0 \times N(d_{12}),\]  

(13)

\[d_{11} = \frac{\ln(X_0 / Q_0) + (\mu + \sigma^2 / 2)T}{\sigma \sqrt{T}},\]  

(14)

\[d_{12} = d_{11} - \sigma \sqrt{T} = \frac{\ln(X_0 / Q_0) + (\mu - \sigma^2 / 2)T}{\sigma \sqrt{T}}.\]  

(15)

(2) \((X_t - Q_1) \times P(X_t \geq Q_1)\)

The calculation of this component begins with

\[(X_t - Q_1) \times P(X_t \geq Q_1) = \int_{Q_1}^{\infty} (X_t - Q_1) \times f(X_t) \, dx \, \exp(-\theta),\]  

(16)

and, after a similar algebraic procedure, concludes also with a close-form outcome for second component of Eq. (11):

\[C_2 = X_0 \exp(\mu T) \times N(d_{21}) - Q_0 \times N(d_{22}),\]  

(17)

\[d_{21} = \frac{\ln(X_0 / Q_1) + (\mu + \sigma^2 / 2)T}{\sigma \sqrt{T}},\]  

(18)

\[d_{22} = d_{21} - \sigma \sqrt{T} = \frac{\ln(X_0 / Q_1) + (\mu - \sigma^2 / 2)T}{\sigma \sqrt{T}}.\]  

(19)

**Right-hand Third and Fourth Terms**

(1) Third Term

\[(c - v)(Q_0 - X_t) \times P(Q_0 \leq X_t \leq Q_0)\]

\[= (c - v)Q_0 \times P(X_t \leq Q_0) - (c - v)Q_t \times P(X_t \leq Q_0)\]  

(20)

(2) Fourth Term

\[(c - v)(Q_1 - X_t) \times P(X_t \leq Q_1)\]

\[= (c - v)Q_1 \times P(X_t \leq Q_1) - (c - v)Q_t \times P(X_t \leq Q_1)\]  

(21)

Combining Eqs. (20) and (21) can get:

\[(c - v)(Q_0 - X_t) \times P(X_t < Q_0) - (c - v)(Q_1 - X_t) \times P(X_t < Q_1)\]

\[= (c - v)(P - P_1).\]  

(22)

It is also evident that Eq. (22) is equivalent to be composed of two put options with which the exercise prices are \(Q_0\) and \(Q_1\), respectively.

The deduction process for these two terms is similar to that for the above first and second terms, and hence, for brevity, the detailed course is not described here and only the resulting solutions are presented.

(1) \((Q_0 - X_t) \times P(X_t < Q_0)\)

The calculation begins with

\[(Q_0 - X_t) \times P(X_t < Q_0) = \int_{Q_0}^{0} (Q_0 - X_t) \times f(X_t) \, dx \, \exp(-\theta).\]  

(23)

Lastly, the solution for the component can be derived by Eq. (23) as Eq. (24) shows:

\[P_1 = Q_0 \times N(-d_{31}) - X_0 \exp(\mu T) \times N(-d_{32}),\]  

(24)

\[d_{31} = \frac{\ln(X_0 / Q_0) + (\mu + \sigma^2 / 2)T}{\sigma \sqrt{T}},\]  

(25)

\[d_{32} = d_{31} - \sigma \sqrt{T} = \frac{\ln(X_0 / Q_0) + (\mu - \sigma^2 / 2)T}{\sigma \sqrt{T}}.\]  

(26)

(2) \((Q_1 - X_t) \times P(X_t < Q_1)\)

Similarly, a close-form solution for the component is also worked out through starting with Eq. (27) and ending with Eq. (28) as follows.

\[(Q_1 - X_t) \times P(X_t < Q_1) = \int_{Q_1}^{0} (Q_1 - X_t) \times f(X_t) \, dx \, \exp(-\theta),\]  

(27)

\[P_2 = Q_1 \times N(-d_{41}) - X_0 \exp(\mu T) \times N(-d_{42}),\]  

(28)

\[d_{41} = \frac{\ln(X_0 / Q_1) + (\mu + \sigma^2 / 2)T}{\sigma \sqrt{T}},\]  

(29)

\[d_{42} = d_{41} - \sigma \sqrt{T} = \frac{\ln(X_0 / Q_1) + (\mu - \sigma^2 / 2)T}{\sigma \sqrt{T}}.\]  

(30)

Accordingly, the expected end-of-period value of the option on order quantity (quantity flexibility) can be estimated through coupling Eqs. (13), (17), (24), and (28), as follows:

\[E[B_t] = (p - c) (C_1 - C_2) + (c - v) (P_1 - P_2).\]  

(31)

The premium of the option on order quantity equals to the present value of expected end-of-period value and can be straightforwardly obtained through discounting as follows:

\[E[B_t] = E[B_T] \exp(-rT)\]

\[= \left[(p - c) (C_1 - C_2) + (c - v) (P_1 - P_2)\right] \exp(-rT).\]  

(32)

The rationality and validity of the proposed approach for assessing option on order quantity can be elementally verified by testing the following three situations.
(1) If $Q_2 \geq Q_0$ and $Q_1 \leq Q_0$ then $C_1 \geq C_2$ and $P_1 \geq P_2$, and it follows that $E[B] \geq 0$.

(2) The first case is the buyer entitled to order an arbitrary quantity; that is, the buyer has an order flexibility interval between $Q_i = 0$ and $Q_i \to \infty$ (or say $Q_i \gg Q_0$). In such a case, $C_2$ and $P_2$ are approximately zero. Consequently, the premium for option on order quantity turns out:

$$E[B] = (p - c)C_1 + (c - v)P_1.$$ 

As expected, the buyer clearly pays the highest premium.

(3) If the buyer does not request order flexibility, $Q_i = Q_0$ and $Q_i = Q_0$. Such a case is straightforward: $C_2 = C_1$ and $P_2 = P_1$ such that

$$E[B] = (p - c)(C_1 - C_1) + (c - v)(P_1 - P_1) = 0.$$ 

Therefore, as in a traditional order contract, no additional premium is needed.

**Numerical Analysis**

Table 1 shows the parameter specification for a plausibly designed quantity flexibility contract that is taken as example in the numerical instance. After applying the proposed assessment approach and value settings for main parameters, the assessing results are exhibited in Table 2. Observed from Table 2, 3.4961% of hedging cost ratio is quite close to that of typical options in which hedging cost ratio usually ranges from 2% to 5% and is thus considered rational and acceptable. Besides, because the buyer pays an additional premium of 1,748,047 USD, if actual demand falls beyond the interval (94,173, 117,480) it is relatively profitable for the buyer when the initial premium (hedging cost) is taken into account. This indeed implies that the buyer and seller jointly realize the risk and profit sharing resulted from uncertain future market demand.

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Parameters for Designed Quantity Flexibility Contract in the Numerical Instance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameters</td>
<td>Values</td>
</tr>
<tr>
<td>Actual demand during previous period</td>
<td>$X_0 = 100,000$</td>
</tr>
<tr>
<td>Original order quantity</td>
<td>$Q_0 = 100,000$</td>
</tr>
<tr>
<td>Lower-bound order quantity</td>
<td>$Q_1 = 120,000$</td>
</tr>
<tr>
<td>Upper-bound order quantity</td>
<td>$Q_2 = 80,000$</td>
</tr>
<tr>
<td>Annual expected growth rate of demand</td>
<td>$\mu = 30%$</td>
</tr>
<tr>
<td>Standard deviation of the demand growth rate</td>
<td>$\sigma = 25%$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 2</th>
<th>Assessing Results for the Numerical Instance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Item</td>
<td>Outcome</td>
</tr>
<tr>
<td>Expected end-of-period value of first call option</td>
<td>$C_1 = 18,280$ USD</td>
</tr>
<tr>
<td>Expected end-of-period value of second call option</td>
<td>$C_2 = 6,547$ USD</td>
</tr>
<tr>
<td>Expected end-of-period value of first put option</td>
<td>$P_1 = 2,096$ USD</td>
</tr>
<tr>
<td>Expected end-of-period value of second put option</td>
<td>$P_2 = 107$ USD</td>
</tr>
<tr>
<td>Premium for option on order quantity</td>
<td>$E[B] = 1,748,047$ USD</td>
</tr>
<tr>
<td>Hedging cost ratio</td>
<td>$E[B]/(Q_0c) = 3.4961%$</td>
</tr>
</tbody>
</table>
Next, the sensitivity analysis for main parameters is conducted separately below.

1. **Quantity Flexibility Bound**

Fig. 1 clearly shows that the premiums dramatically increase when lower quantity bounds are somewhat lowered. However, the following increments in premium substantially decrease as lower quantity bounds are sustained decline to around 80,000, and increments almost approximate to nil for these quantities after 65,000. Figs. 2 and 3 respectively show the relative variability between premium and flexibility quantity bound for these two scenarios of only upper bound varying and both lower and upper bound varying. Figs. 2 and 3 show a quite similar outcome, which the premium considerably increases initially, but the increments substantially decrease as lower quantity bound intervals are extended. The presumable reason for observations above is that occurrences of very high and low demand should be unusual; thus, corresponding probabilities are too small to arise.
(2) Demand drift

The profile appeared in Fig. 4 is clearly characterized by an asymmetric convex function, which indicates that the growth rate in demand gradually increases the premium then sizably decreases steadily and until growth rate in demand reaches a minimum of 40%; it then gradually increases with a relatively smaller amount. The higher premium is paid initially because the demand during upcoming selling season is likely below 94,173, which is relatively favorable to buyer, when the growth rate in demand is lesser. With the growth rate in demand increasingly rising enlarges the probability of actual demand falling into the relatively unfavorable interval (94,173, 117,480) for the buyer. In the meantime, this relatively adverse situation returns in the decrease of premium to reflect the reduced expected profit for the buyer. However, the growth rate of demand further increases such that actual demand exceeds 117,480, which turns again into relatively advantageous for the buyer. Thus, the higher premium is required by the seller to compensate for increased risk.

(3) Demand volatility

According to Fig. 5, the premium also transiently declines first and climbs subsequently, and the increments projects somewhat first increasing and then decreasing as demand volatility increases. The underlying reason is also the same as growth rate of demand. At the outset, as volatility of demand getting larger, the actual demand will fall into the unfavorable intervals for buyer with a higher probability. Soon after, the premium reversal occurs at demand volatility of about 15%, and, afterward, advances constantly as volatility of demand further intensifying to reflect the fact, which it has become more and more favorable to buyer.

![Figure 4](image1.png) Relative Changing Relationship between Premium and Demand Drift

![Figure 5](image2.png) Relative Changing Relationship between Premium and Demand Volatility
Figure 6  Relative Changing Relationship between Premium and Original Order Quantity

(4) Original order quantity

An interesting interaction appears in Fig. 6 and is evidently characterized by another convex function, which suggests that more premiums must be paid when the original order quantities are farther (greater or lesser) away from 100,000, especially for greater original order quantities. A noteworthy implication from this finding is that the original order quantity would better select close to the actual demand during previous selling season so as to yield the effectiveness of economy in premium expense. The excessively deviating from original order quantity will lead to an unnecessary consequence of costly premium.

4. Concluding Remarks

Because of the practical application limitations of existing flexibility contracts, this study suggests a distinct scheme for quantity flexibility contracts which are unrestricted in order flexibility selection. Moreover, the quantity flexibility contracts considered here are applicable to established products with high demand variability and uncertainty. The enlargement or diminution of order quantity can be accomplished by implementing stock supplements or leftover returns within the appointed flexibility in due course during the selling season. Among other things, a critical successful factor for the quantity flexibility contracts is to reasonably assessing their intrinsic values to properly compensate the seller’s potential risk. To this end, this study demonstrates that quantity flexibility contracts are comparable to options on order quantity. Thus, a real options approach is used here to develop a logical and accurate approach for modeling and assessing quantity flexibility contracts. The model validity is elementally confirmed by testing against extreme cases. Finally, the proposed approach is illustrated and verified by conducting a numerical instance. The rationality can be ascertained in terms of the conceivable hedging cost ratio, which is consistent with general options.

REFERENCES


